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coordinates

UNIVERSITY OF PENNSYLVANIA

A GENERALIZED EQUATION OF THE VIBRATING MEMBRANE EXPRESSED IN CURVILINEAR COORDINATES

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1. GENERAL EQUATION OF THE VIBRATING MEMBRANE.

The problem of the vibrating membrane has been attacked by previous writers from the standpoint of the shape of the boundaries of the membrane, rectangular, circular, parabolic, elliptic. No attempt has been made to derive a single equation which contains all of these. Here we shall derive such an equation by the use of a general system of curvilinear coördinates. These coördinates shall be restricted only by the condition that the resulting equation shall have a solution in the form of a product of functions of the independent variables involved, each function depending only upon a single variable.

If a stretched elastic membrane is fastened at the edges and is made to vibrate at right angles to the plane of the membrane, its equation of motion in rectangular coördinates is,

$$\frac{\partial^2 V}{\partial t^2} = a^2\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) \tag{1}$$

where a is a physical constant depending upon the mass per unit area and the tension along any line in the surface of the membrane. In (1) put $V = T \cdot u$, where T is a function of t only and u a function of x and y only. The equation then separates into the two parts,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + K^2 u = 0, \qquad \text{and} \qquad \frac{dT^2}{dt^2} + a^2K^2T = 0. \tag{2}$$

where K is any constant. Solutions of the latter are $T = \sin aKt$ and $T = \cos aKt$.

Now suppose x and y are determined as functions of two new unrestricted variables, $x = f_1(\alpha, \beta)$, $y = f_2(\alpha, \beta)$ and express $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$ in terms of these new variables,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \alpha^2}\left[\left(\frac{\partial \alpha}{\partial x}\right)^2 + \left(\frac{\partial \alpha}{\partial y}\right)^2\right] + \frac{\partial^2 u}{\partial \beta^2}\left[\left(\frac{\partial \beta}{\partial x}\right)^2\right.$$

$$(3) \qquad + \left(\frac{\partial \beta}{\partial y}\right)^2 \Bigg] + \frac{2\partial^2 u}{\partial\alpha\partial\beta}\left[\frac{\partial \alpha}{\partial x}\cdot\frac{\partial \beta}{\partial x} + \frac{\partial \alpha}{\partial y}\cdot\frac{\partial \beta}{\partial y}\right]$$

$$+ \frac{\partial u}{\partial \alpha}\left[\frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2}\right] + \frac{\partial u}{\partial \beta}\left[\frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^2 \beta}{\partial y^2}\right].$$

Assume now that

$$(4) \qquad \begin{aligned} x + iy &= f(\alpha + i\beta), \\ x - iy &= f(\alpha - i\beta). \end{aligned}$$

This assumption will greatly simplify (3). Putting this simplified result in (2) we obtain,

$$(5) \qquad \left[\left(\frac{\partial \alpha}{\partial x}\right)^2 + \left(\frac{\partial \alpha}{\partial y}\right)^2\right]\cdot\left[\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2}\right] = -K^2 u.$$

By means of the orthogonal relation

$$\left(\frac{\partial \alpha}{\partial x}\right)^2 + \left(\frac{\partial \alpha}{\partial y}\right)^2 = 1 \div \left[\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2\right]$$

this equation becomes

$$(6) \qquad \frac{\partial^2 u}{\partial^2 \alpha} + \frac{\partial^2 u}{\partial \beta^2} = -K^2 u\left[\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2\right].$$

The right member of this may be written in a more convenient form. From (4) we may derive the relation,

$$(7) \qquad \left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 = f'(\alpha + i\beta)\cdot f'(\alpha - i\beta)$$

which when put in (6) gives,

$$(8) \qquad \frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} = -K^2 u\cdot f'(\alpha + i\beta)\cdot f'(\alpha - i\beta).$$

On the assumption that this equation has a solution of the form $u = G(\alpha)\cdot H(\beta)$, it is necessary that the relation (9) should hold.

$$(9) \qquad f'(\alpha + i\beta)\cdot f'(\alpha - i\beta) = E(\alpha) + F(\beta)$$

where E and F are functions which we shall determine now. By differentiating first with respect to α and then with respect to β we obtain the equation

$$(10) \quad f'''(\alpha + i\beta)\cdot f'(\alpha - i\beta) - f'''(\alpha - i\beta)\cdot f'(\alpha + i\beta) = 0.$$

This equation has the solutions

(11) $f'(\alpha + i\beta) = B \sin A(\alpha + i\beta)$ and $C \cos A(\alpha + i\beta)$,

A, B, C are constants.

These are the functions then which satisfy the assumption in (9). Substituting in (9) we obtain,

$$E(\alpha) + F(\beta) = B^2 \sin A(\alpha + i\beta) \cdot \sin A(\alpha - i\beta)$$

$$= B^2(\sin^2 A\alpha + \sinh^2 A\beta)$$

and (8) becomes,

$$\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} + K^2 B^2 u(\sinh^2 A\beta + \sin^2 A\alpha) = 0.$$

Hence (1) may now be written,

$$[B^2(\sinh^2 A\beta + \sin^2 A\alpha)] \frac{\partial^2 V}{\partial t^2} = a^2 \left(\frac{\partial^2 V}{\partial \alpha^2} + \frac{\partial^2 V}{\partial \beta^2} \right). \tag{12}$$

That the assumption in (9) really has resulted in changing our curvilinear coördinate system to families of orthogonal confocal ellipses and hyperbolas may be seen from (11) and (4). And the method here employed shows that (12) is the most general equation obtainable for the vibrating membrane which has for its solution the function $V = T(t) \cdot G(\alpha) \cdot H(\beta)$.

2. DERIVATION AND DISCUSSION OF THE GENERAL EQUATIONS OF THE CONFOCAL CONICS USED AS COÖRDINATE SYSTEMS.

From the equations of the families of the curves of the orthogonal curvilinear coördinate system in which (12) is expressed, we purpose now to derive the following systems as special cases,

(*a*) System of rectilinear lines.
(*b*) System of concentric circles and radial lines.
(*c*) System of confocal parabolas.

In order to do this, however, care must be exercised in selecting a general solution for (10). Take this solution in the form

$$f'(\alpha + i\beta) = B \sin A(\alpha + i\beta) - C \cos A(\alpha + i\beta).$$

The system of rectilinear lines obtained in (a) may also be derived from the system of confocal parabolas. Put $\bar{A}\beta^2 = p$ in (22) and $\bar{A}\alpha^2 = q$ in (23) and change the origin to the point $(a, 0)$.

$$y^2 = 2p(x + a) + p^2, \tag{24}$$

$$y^2 = -2q(x + a) + q^2. \tag{25}$$

Now if we let $2pa = m^2$ in (24) and then let $a = \infty$ in the negative direction while m remains finite, we obtain

$$y = \pm m, \tag{26}$$

and this condition can be brought about by having $\beta \doteq 0$ as $a = \infty$ in such a way that $2\bar{A}\beta^2 \cdot a = m^2$, for $\bar{A}$ is finite.

If we let $a - q/2 = -n$ in (25) and then let $a = \infty$ while n remains finite,

$$x = n. \tag{27}$$

And this condition can be brought about by having $\alpha = \infty$ as $a = \infty$ in such a way that $(\bar{A}\alpha^2/2) - a$ always remains finite. For those parabolas of the family in (23) in which $(\bar{A}\alpha^2/2) > a$ the parameter n is positive and for those in which $(\bar{A}\alpha^2/2) < a$ the parameter n will be negative.

If we had changed the origin of the curves in (14) and (15) to the right focus and then sent the left to infinity in the negative direction, we would have obtained

$$y^2 = \bar{A}\alpha^2(\bar{A}\alpha^2 + 2x) \qquad \text{and} \qquad y^2 = \bar{A}\beta^2(\bar{A}\beta^2 - 2x)$$

as the equations of the families of parabolas. From these the families of rectilinear lines may be obtained by changing the origin to the point $(-a, 0)$ and then sending the focus to infinity in the positive direction.

3. DISCUSSION OF THE GENERAL DIFFERENTIAL EQUATION OF THE VIBRATING MEMBRANE.

The same values of the constants used for the derivation of the three special systems of curvilinear coördinates from the general system should, when put in the general equation of the

vibrating membrane, with the time element removed, give three special equations expressed in the above mentioned coördinates.

With the aid of (13) equation (6) may be written

$$(28)\quad \frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} + K^2C^2A^2u[\cosh^2(A\beta + D) - \cos^2(A\alpha + B)] = 0.$$

This separates into,

$$(29)\quad \frac{d^2Y}{d\beta^2} + [K^2C^2A^2\cosh^2(A\beta + D) - M^2]Y = 0,$$

$$(30)\quad \frac{d^2X}{d\alpha^2} - [K^2C^2A^2\cos^2(A\alpha + B) - M^2]X = 0.$$

The family of horizontal lines, $y = \pm F\beta$, were obtained from the family of confocal ellipses by having $C = \infty$ and $A \doteq 0$ in such a manner that $C \cdot A = F$ and $D = 0$. If these values are put in (29), it becomes

$$(31)\quad \frac{d^2Y}{dy^2} + (K^2 - \overline{M}^2)Y = 0 \qquad \text{where} \qquad \overline{M} = \frac{M}{F}.$$

The family of vertical lines $x = \mp F\alpha$ were obtained from the family of confocal hyperbolas by having $C = \infty$ and $A \doteq 0$ in such a way that $C \cdot A = F$ and $B = \pi/2$. If these values are put in (30) it becomes

$$(32)\quad \frac{d^2X}{dx^2} + \overline{M}^2X = 0 \qquad \text{where} \qquad \overline{M} = \frac{M}{F}.$$

And these are the equations into which

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + K^2u = 0$$

separates.

The family of concentric circles, $x^2 + y^2 = F^2e^{2AB}$, were obtained from the family of confocal ellipses by having $D = \infty$ and $C \doteq 0$ in such a way that $C \cosh D = F$ and $C \sinh D = F$. Moreover

$$\frac{d^2Y}{d\beta^2} = A^2r^2\frac{d^2Y}{dr^2} + A^2r\frac{dY}{dr}$$

in this instance. If these values are put in (29) it becomes

$$\frac{d^2Y}{dr^2}+\frac{1}{r}\frac{dY}{dr}+\left(\bar{K}^2-\frac{\bar{M}^2}{r^2}\right)Y=0 \qquad \text{where} \tag{33}$$

$$\bar{K}=KF \qquad \text{and} \qquad \bar{M}=\frac{M}{A}.$$

The family of radial lines, $y = \pm x \tan A\alpha$, were obtained from the family of confocal hyperbolas by having $C \doteq 0$ and $B = 0$. If these together with

$$\varphi = \tan^{-1}\frac{y}{x} = A\alpha$$

are put in (29) it becomes

$$\frac{d^2X}{d\varphi^2}+\bar{M}^2X=0 \qquad \text{where} \qquad \bar{M}=\frac{M}{A}. \tag{34}$$

And these are the equations into which the equation which defines the function u in circular coördinates,

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}+K^2u=0,$$

separates.

It will simplify matters somewhat, in obtaining the equations in parabolic coördinates, if we adopt the equation

$$\frac{\partial^2 u}{\partial \alpha^2}+\frac{\partial^2 u}{\partial \beta^2}+K^2C^2A^2u[\sinh^2(A\beta+D)+\sin^2(A\alpha+B)]=0 \tag{35}$$

in place of (28). This separates into

$$\frac{d^2Y}{d\beta^2}+[K^2C^2A^2\sinh^2(A\beta+D)-M^2]Y=0, \tag{36}$$

$$\frac{d^2X}{d\alpha^2}+[K^2C^2A^2\sin^2(A\alpha+B)+M^2]X=0. \tag{37}$$

The family of confocal parabolas, $y^2=\bar{A}\beta^2(\bar{A}\beta^2+2x)$, were obtained from the family of confocal ellipses by having $A \doteq 0$ and $C=\infty$ in such a way that $C\cdot A^2=\bar{A}$ and $D=0$. If these values are put in (36) it becomes

$$\frac{d^2Y}{d\beta^2}+(k^2\beta^2-M^2)Y=0, \qquad \text{where} \qquad k^2=K^2\cdot\bar{A}^2. \tag{38}$$

The family of confocal parabolas, $y^2 = \bar{A}\alpha^2(\bar{A}\alpha^2 - 2x)$ were obtained from the family of confocal hyperbolas by using the same values of the constants as above and $B = 0$. If these are put in (37) it becomes

$$(39) \quad \frac{d^2X}{d\alpha^2} + (k^2\alpha^2 + M^2)X = 0 \qquad \text{where} \qquad k^2 = K^2 \cdot \bar{A}^2.$$

4. SOLUTION OF THE EQUATION IN RECTANGULAR AND CIRCULAR COÖRDINATES.

The complete solution of the equation of a stretched elastic rectangular vibrating membrane is obtained by an extension of Fourier's Theorem and may be found in the works of any of the writers on Partial Differential Equations. The complete solution representing the vibrations of a circular membrane is obtained by the use of Bessel's Functions. If the membrane vibrates so that it has circular symmetry about the axis perpendicular to the plane of the boundary at the origin, the J_0 functions furnish the solution. If the mode of vibration is a function of both the coördinates r and φ then the J_n functions furnish the desired solution. And a discussion of the problem may be found in any of the works on Bessel's Functions. What we believe, however, to be a contribution to the literature of Bessel's Functions, is a method of obtaining the following fundamental relations between these functions:

$$(a) \quad \frac{dJ_n(x)}{dx} = \frac{n}{x} J_n(x) - J_{n+1}(x),$$

$$(b) \quad 2\frac{dJ_n(x)}{dx} = J_{n-1}(x) - J_{n+1}(x),$$

$$(c) \quad \frac{dJ_n(x)}{dx} = J_{n-1}(x) - \frac{n}{x} J_n(x),$$

$$(d) \quad \frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x),$$

$$(e) \quad \frac{d^2[\sqrt{x}J_n(x)]}{dx^2} = \left(\frac{4n^2 - 1}{4x^2} - 1\right)\sqrt{x}J_n(x).$$

The usual method of obtaining these relations is to consider the functions entirely independent of the equation of which they are solutions. Since the functions are defined by a differential equation it was thought that the equation itself ought to yield these fundamental relations.

Starting with $J_n(x)$ as a solution of Bessel's Equation,

$$\frac{d^2z}{dx^2}+\frac{1}{x}\frac{dz}{dx}+\left(1-\frac{n^2}{x^2}\right)z=0 \tag{40}$$

and substituting $z = x^n v$, we obtain $v = x^{-n}J_n(x)$ as a solution of

$$\frac{d^2v}{dx^2}+\frac{2n+1}{x}\frac{dv}{dx}+v=0. \tag{41}$$

Substituting this value or v in (41)

$$\frac{d^2}{dx^2}(x^{-n}J_nx)+\frac{2n+1}{x}\frac{d}{dx}(x^{-n}J_nx)+x^{-n}J_nx=0. \tag{42}$$

The corresponding equation for $J_{n+1}x$ is

$$\frac{d^2}{dx^2}(x^{-n-1}J_{n+1}x)+\frac{2n+3}{x}\frac{d}{dx}(x^{-n-1}J_{n+1}x) + x^{-n-1}J_{n+1}x=0. \tag{43}$$

Differentiate (42) with respect to x, add to (43) multiplied by x and arrange terms,

$$\begin{aligned}&\frac{d^2}{dx^2}\left[\frac{d}{dx}(x^{-n}J_nx)+x^{-n}J_{n+1}x\right]\\ &\quad+\frac{2n+1}{x}\frac{d}{dx}\left[\frac{d}{dx}(x^{-n}J_nx)+x^{-n}J_{n+1}x\right]\\ &\quad+\left(1-\frac{2n+1}{x^2}\right)\cdot\left[\frac{d}{dx}(x^{-n}J_nx)+x^{-n}J_{n+1}x\right]=0.\end{aligned} \tag{44}$$

A solution for this is

$$\frac{d}{dx}(x^{-n}J_nx)+x^{-n}J_{n+1}x=0. \tag{45}$$

A corresponding relation between J_nx and $J_{n-1}x$ may be obtained. Put $z = x^{-n}v$ in (40)

$$\frac{d^2v}{dx^2} - \frac{2n-1}{x}\frac{dv}{dx} + v = 0. \tag{46}$$

A solution for this is $v = x^n J_n x$. Put this in (46),

$$\frac{d^2(x^n J_n x)}{dx^2} - \frac{2n-1}{x}\frac{d(x^n J_n x)}{dx} + x^n J_n x = 0. \tag{47}$$

The corresponding equation for $J_{n-1}x$ is

$$\frac{d^2(x^{n-1} J_{n-1} x)}{dx^2} - \frac{2n-3}{x}\frac{d(x^{n-1} J_{n-1} x)}{dx} + x^{n-1} J_{n-1} x = 0. \tag{48}$$

Differentiate (47) with respect to x, subtract (48) multiplied by x from it and arrange terms

$$\begin{aligned} &\frac{d^2}{dx^2}\left[\frac{d}{dx}(x^n J_n x) - x^n J_{n-1} x\right] \\ &\qquad - \frac{2n-1}{x}\frac{d}{dx}\left[\frac{d}{dx}(x^n J_n x) - x^n J_{n-1} x\right] \\ &\qquad + \left(1 + \frac{2n-1}{x^2}\right)\left[\frac{d}{dx}(x^n J_n x) - x^n J_{n-1} x\right] = 0. \end{aligned} \tag{49}$$

A solution for this is

$$\frac{d}{dx}(x^n J_n x) - x^n J_{n-1} x = 0. \tag{50}$$

Relations (*a*), (*b*), (*c*), (*d*) may be obtained from (45) and (50). Relation (*e*) may be obtained from (40) by the substitution, $u = v/\sqrt{x}$, where $v = \sqrt{x} J_n x$.

5. SOLUTION OF THE EQUATION IN PARABOLIC COÖRDINATES.

The equation of the vibrating membrane written in parabolic coördinates is

$$(\alpha^2 + \beta^2)\frac{\partial^2 V}{\partial t^2} = a^2\left(\frac{\partial^2 V}{\partial \alpha^2} + \frac{\partial^2 V}{\partial \beta^2}\right). \tag{51}$$

This separates into

$$\frac{d^2 T}{dt^2} + k^2 T = 0$$

and

$$\frac{d^2X}{d\alpha^2} + (k^2\alpha^2 + \lambda)X = 0, \tag{52}$$

$$\frac{d^2Y}{d\beta^2} + (k^2\beta^2 - \lambda)Y = 0. \tag{53}$$

Dr. H. Weber in his Partiellen Differentialgleichungen, Vol. 2, page 256, finds two imaginary solutions for (52) as limiting forms of the hypergeometric series. He also finds two real solutions in definite integral form. Several new solutions for special values of λ will now be obtained.

Repeated differentiation of (52) will point the way to the construction of equations which when differentiated $1, 2, 3 \cdots n$ times will give (52). The nth equation of this series can be solved if λ has the special value $\lambda = ik(2n + 1)$, where n is a positive integer. And from this solution a solution of (52) may be obtained. It is

$$X = C(\sqrt{e})^{ik\alpha^2} \cdot \frac{d^n}{d\alpha^n}(e^{-ik\alpha^2}). \tag{54}$$

From this a series of functions depending upon n may be obtained. For $n = 0$ we have the very simple one,

$$X = C(\sqrt{e})^{-ik\alpha^2}. \tag{55}$$

Weber's two solutions for $\lambda = ik$ become

$$X_1 = C_1(\sqrt{e})^{ik\alpha^2} \cdot \left[1 + (-ik\alpha^2) + \frac{1}{1\cdot 2}(-ik\alpha^2)^2 + \frac{1}{1\cdot 2\cdot 3}(-ik\alpha^2)^3 + \cdots\right], \tag{56}$$

$$X_2 = C_2(\sqrt{e})^{ik\alpha^2}\sqrt{(-ik\alpha^2)}\left[1 + \frac{2}{1\cdot 3}(-ik\alpha^2) + \frac{4}{1\cdot 3\cdot 5}(-ik\alpha^2)^2 + \frac{8}{1\cdot 3\cdot 5\cdot 7}(-ik\alpha^2)^3 \cdots\right]. \tag{57}$$

Equation (56) reduces to $X_1 = C_1(\sqrt{e})^{-ik\alpha^2}$ and this agrees with (55). The series in (57) may be summed by means of the integral

$$\int_0^x e^{-x}(\sqrt{x})^{-1}dx = 2e^{-x}\sqrt{x}\left[1 + \frac{2}{1\cdot 3}x + \frac{4}{1\cdot 3\cdot 5}x^2 + \cdots\right],$$

which is obtained by repeated integration by parts of the left member. Applying this to (57) we obtain,

$$(58)\qquad X_2 = (\sqrt{e})^{-ik\alpha^2}(\sqrt{-ik})\int_0^{-ik\alpha^2} e^{ik\alpha^2}d\alpha.$$

None of the above results lend themselves to the solution of the problem of the vibrating membrane because they are imaginary. For $\lambda = 0$, however, two real solutions of (52) are

$$(59)\quad X_1 = a_0\left[1 - \frac{k^2\alpha^4}{3\cdot 4} + \frac{k^4\alpha^8}{3\cdot 7\cdot 4\cdot 8} - \frac{k^6\alpha^{12}}{3\cdot 7\cdot 11\cdot 4\cdot 8\cdot 12} + \cdots\right],$$

$$(60)\quad \begin{aligned} X_2 = a_1(\sqrt{k\alpha^2})\Bigg[1 - \frac{k^2\alpha^4}{4\cdot 5} &+ \frac{k^4\alpha^8}{4\cdot 8\cdot 5\cdot 9} \\ &- \frac{k^6\alpha^{12}}{4\cdot 8\cdot 12\cdot 5\cdot 9\cdot 13} + \cdots\Bigg]. \end{aligned}$$

The corresponding Y-functions for (58) may be found from these by replacing α with β. These functions then may be used in solving the problem of the vibrating membrane with parabolic boundaries when the mode of vibration is such that $\lambda = 0$.

The first two roots of $X_1 = 0$ and $X_2 = 0$ in (59) and (60) have been computed by the writer. They are

$$\text{for } X_1 = 0\begin{cases} x_1 = 4.013\cdots, \\ x_2 = 10.246\cdots, \end{cases}\qquad \text{for } X_2 = 0\begin{cases} x_1 = 5.563\cdots, \\ x_2 = 11.818\cdots. \end{cases}$$

The roots for $Y_1 = 0$ and $Y_2 = 0$ are identical with the corresponding roots of $X_1 = 0$ and $X_2 = 0$. These roots are the values of the parameter which give the nodal lines for this particular mode of vibration.

Weber's two imaginary solutions for $\lambda = 0$ are,

$$(61)\quad \begin{aligned} X_1 = (\sqrt{e})^{ik\alpha^2}\Bigg[1 + \tfrac{1}{4}(-ik\alpha^2) &+ \frac{1\cdot 5}{4\cdot 3\cdot 2!}(-ik\alpha^2)^2 \\ &+ \frac{1\cdot 3\cdot 13}{7\cdot 4^2\cdot 4!} - (-ik\alpha^2)^3 + \cdots\Bigg], \end{aligned}$$

$$(62)\quad \begin{aligned} X_2 = (\sqrt{e})^{ik\alpha^2}\sqrt{(-ik\alpha^2)}\Big[1 &+ \tfrac{1}{2}(-ik\alpha^2) \\ &+ \frac{7}{5\cdot 4\cdot 2!}(-ik\alpha^2)^2 + \frac{11}{5\cdot 2^3\cdot 3!}(-ik\alpha^2)^3\cdots\Big]. \end{aligned}$$

Equation (61) reduces to (59), a real solution, if $a_0 = 1$. Equation (62) reduces to (60), a real solution, if $a_1 = \sqrt{-i}$. These results may be obtained by performing the indicated multiplications in each case.

6. REVIEW OF THE WORK PREVIOUSLY DONE ON THE SOLUTION OF THE EQUATION IN PARABOLIC AND ELLIPTIC COÖRDINATES.

Parabolic coördinates: The real solutions of the equation

$$\frac{D^2X}{d\alpha^2} + (k^2\alpha^2 + \lambda)X = 0$$

obtained by Dr. H. Weber* and referred to previously are

$$(63)\quad X_1 = \int_0^1 s^{-\frac{3}{4}}(1-s)^{-\frac{3}{4}}\cos\left[k^2\alpha^2(s-\tfrac{1}{2}) + \frac{\lambda}{4k}\log\frac{s}{1-s}\right]ds,$$

$$(64)\quad X_2 = \int_0^1 s^{-\frac{1}{4}}(1-s)^{-\frac{1}{4}}\cos\left[k^2\alpha^2(s-\tfrac{1}{2}) + \frac{\lambda}{4k}\log\frac{s}{1-s}\right]ds.$$

The corresponding functions of β may be obtained from these by changing α into β and λ into $-\lambda$. Using these solutions Weber† discusses the vibration of regions bounded by parabolic nodal lines. He proves the theorem, "If a function of u different from zero, defined by the equation

$$\frac{\partial^2 u}{\partial\alpha^2} + \frac{\partial^2 u}{\partial\beta^2} + k^2u = 0$$

is continuous and its first differential quotient also, inside a bounded region but vanishes on the boundary, then k^2 must be real and positive." Such regions are of three kinds,

* "Partielle Differentialgleichungen," Vol. 2, page 256.

† *Mathematische Annalen*, Vol. 1, 1868.

(*a*) Bounded by two parabolas one from each family.

(*b*) Bounded by two parabolas from one family and one from the other.

(*c*) Bounded by two parabolas from one family and two from the other.

For (*a*) the boundary conditions are $X = 0$ for $\alpha = \pm \alpha_1$ and $Y = 0$ for $\beta = \pm \beta_1$. Under these conditions four particular solutions may be found, $u = X_1 \cdot Y_1$, $u = X_2 \cdot Y_2$, $u = X_1 \cdot Y_2$, $u = X_2 \cdot Y_1$. The last two however will not satisfy the condition that u shall be single valued within the region. Two separate systems of solutions then may be built from $u = X_1 \cdot Y_1$ and $u = X_2 \cdot Y_2$. In each solution the constants λ and k appear and may be determined as the roots of the two transcendental equations involved.

For (*b*) the boundary conditions are $X = 0$ for $\alpha = \alpha_1$, $\alpha = \alpha_2$ and $Y = 0$ for $\beta = \pm \beta_1$. Two types of solutions will fit these conditions, $u = Y_1(A_1X_1 + A_2X_2)$, $u = Y_2(A_1X_1 + A_2X_2)$. These give rise to two systems of solutions. The constants in each solution are A_1/A_2, k, λ and they may be determined from the three transcendental equations involved.

For (*c*) the boundary conditions are $X = 0$ for $\alpha = \alpha_1$, $\alpha = \alpha_2$ and $Y = 0$ for $\beta = \beta_1$, $\beta = \beta_2$. One system of solutions only can be used here, $u = (A_1X_1 + A_2X_2)(B_1Y_1 + B_2Y_2)$ and there are four transcendental equations to determine the constants A_1/A_2, B_1/B_2, λ, k.

M. Hartenstein* obtains solutions for the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$$

in parabolic coördinates for negative values of k^2 by transforming the solutions of the equation in circular coördinates (Bessel's Functions) into their corresponding functions expressed in parabolic coördinates. This is a part of a general discussion of the equation expressed in the four systems of coördinates used in the third part of this paper. The results obtained contribute nothing to the discussion of the problem of the vibrating membrane.

* *Archiv der Mathematik und Physik* (2), Vol. 14.

Elliptic coördinates: The equation of motion of the vibrating membrane expressed in elliptic coördinates is,

(65) $$(\cosh^2\beta - \cos^2\alpha)\frac{\partial^2 V}{\partial t^2} = a^2\left(\frac{\partial^2 V}{\partial \alpha^2} + \frac{\partial^2 V}{\partial \beta^2}\right)$$

and separates into

$$\frac{d^2 T}{dt^2} + a^2k^2T = 0$$

and

(66) $$\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} + k^2(\cosh^2\beta - \cos^2\alpha)u = 0.$$

This last equation separates into,

(67) $$\frac{d^2 X}{d\alpha^2} - (k^2\cos^2\alpha - \lambda)X = 0,$$

(68) $$\frac{d^2 Y}{d\beta^2} + (k^2\cosh^2\beta - \lambda)Y = 0.$$

Dr. Heine* shows that (66) is a limiting form of the differential equation of the Lamé Functions in the same way as the differential equation of the Bessel's Function is a limiting form of the differential equation of the Spherical Functions. He solves (67) by assuming that the constant λ depends in a definite way (as a root of a known transcendental equation) on the constant k and the solutions periodic functions of α. Four classes of functions are found to be solutions of (67) under these assumptions. They are,

(*a*) Series of cosines in ascending even multiples of α.
(*b*) Series of cosines in ascending odd multiples of α.
(*c*) Series of sines in ascending odd multiples of α.
(*d*) Series of sines in ascending even multiples of α.

E. Mathieu† in discussing the vibratory movement of an elliptic membrane, uses periodic functions as solutions also of the above equations. He shows that the general solution of such linear differential equations of the second order as (67) and (68) is composed of two parts, one zero for the zero value of the argument and the other a maximum. If the constant λ has

* "Handbuch der Kuglefunctionen," page 401.
† *Journal de Liouville* (2), Vol. 13, 1886.

the form $\lambda = g^2 + bk^4 + ck^6 + \cdots$ where g is an integer and b, c, etc. constants determined by the periodic property, the solutions of (67) and (68) may be written as periodic functions of α and β in ascending powers of k^2, and they become zero g times from 0 to π. For the special case when $k = 0$ these solutions become $X_1 = \sin(g\alpha)$, $X_2 = \cos(g\alpha)$, $Y_1 = \sin(gi\beta)$, $Y_2 = \cos(gi\beta)$. Solutions for these equations are also given in series according to the ascending powers of sin and cos of α and β. In this case the general solution of (67) is the sum of solutions one even and the other odd in sin α and one even and the other odd in cos α. These results agree with those given by Heine. From these solutions two types of vibration are possible. And some simple cases of hyperbolic nodal lines are, the major axis alone, both together, two asymptotes of the same hyperbola, two asymptotes of the same hyperbola with the major and minor axes. The elliptic nodal lines are obtained from $Y_1 = 0$ and $Y_2 = 0$ as follows: If $\beta = B$ on the boundary of the membrane when $Y = 0$, the constant λ may be found from $Y(B, \lambda) = 0$. If $\lambda_1, \lambda_2, \lambda_3, \cdots$ are the roots of this equation in order of increasing magnitude, $Y(\beta, \lambda_n) = 0$ will give, through its roots in β, the parameters of the elliptic nodal lines. This equation is shown to have $n - 1$ roots less than B, and therefore the number of elliptic nodal lines is $n - 1$.

F. Pockels* has investigated the function defined by (66) for the region bounded by any two ellipses, $\beta = \beta_1$ and $\beta = \beta_2$ and any two hyperbolas, $\alpha = \alpha_1$ and $\alpha = \alpha_2$ under the supposition that $u = 0$ on all four sides. He proves the following theorem: "There is for such a region a doubly infinite series of normal functions which satisfy the boundary condition $u = 0$ on all four sides, and of which a definite number $(m - 1)$ of elliptic and a definite number $(n - 1)$ of hyperbolic nodal lines are characteristic, $(m = 1, 2, \cdots, \infty, n = 1, 2, \cdots, \infty)$." By cutting the bounded region along the line connecting the foci, and assuming that u and its first derivative are continuous along this new boundary, the new region thus formed may be regarded as a simply connected Riemann surface. This makes possible the consideration of regions bounded by less than four curves, as

* "Ueber die Partielle Differentialgleichung $\Delta u + k_2 u = 0$."